Mistake in a Vector Problem

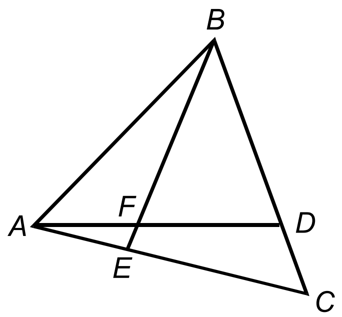
Question

ABC is a triangle. The position vectors of *A*, *B* and *C* with respect to *O* are **a**, **b** and **c** respectively. *D* and *E* are points on *BC* and *AC* respectively so that *BD* : *DC* = *CE* : *EA* = 2 : 1. *F* is the intersection of *AD* and *BE*. Let $\overrightarrow{OF} = \boldsymbol{f}$, *AF* : *FD* = 1 : *r* and *BF* : *FE* = *s* : 1.

(a) Express ***f*** in terms of **a**, **b**, **c** and *r*.

(b) Express ***f*** in terms of **a**, **b**, **c** and *s*.

(c) Hence find *r* and *s*.



Proposed Solution

$$\overrightarrow{OD} = \frac{\mathbf{b}+2\mathbf{c}}{3}, \quad \overrightarrow{OE} = \frac{2\mathbf{a}+\mathbf{c}}{3}$$

(a) $$\vec{f} = \frac{\overrightarrow{OD}+r\mathbf{a}}{1+r} = \frac{\frac{b+2c}{3}+r\mathbf{a}}{1+r} = \underline{\underline{\frac{r}{1+r}\mathbf{a} + \frac{1}{3(1+r)}\mathbf{b} + \frac{2}{3(1+r)}\mathbf{c}}}$$

(b) $$\vec{f} = \frac{s\overrightarrow{OE}+\mathbf{b}}{1+s} = \frac{\frac{2sa+sc}{3}+\mathbf{b}}{1+s} = \underline{\underline{\frac{2s}{3(1+s)}\mathbf{a} + \frac{1}{1+s}\mathbf{b} + \frac{s}{3(1+s)}\mathbf{c}}}$$

(c) From **(a)** and **(b)**, $\frac{r}{1+r} = \frac{2s}{3(1+s)}$......(1) $\quad \frac{1}{3(1+r)} = \frac{1}{1+s}$..........(2) $\quad \frac{2}{3(1+r)} = \frac{s}{3(1+s)}$......(3)

From(2), $1+s = 3+3r$

$s = 2+3r$............................(4)

Substitute $s = 2 + 3r$ into (1),

$$\frac{r}{1+r} = \frac{2(2+3r)}{3(3+3r)} \Rightarrow \frac{r}{1+r} = \frac{4+6r}{9(1+r)} \Rightarrow 9r = 4+6r \Rightarrow r = \underline{\underline{\frac{4}{3}}}$$

Substitute $r = \frac{4}{3}$ into (4), $\quad s = 2+3(\frac{4}{3}) = \underline{\underline{6}}$

Analysis

In planar vectors,

$x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$ does not imply $x_1 = y_2$ and $x_2 = y_2$ and $x_3 = y_3$,

even if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-zero, non-parallel vectors.

This can be seen from the counter-example : $\mathbf{i}+\mathbf{j}+(\mathbf{i}+\mathbf{j}) = 3\mathbf{i}+3\mathbf{j}-(\mathbf{i}+\mathbf{j})$

Part **(c)** should be done like this:

Since the origin O is rather arbitrary, we choose A to be our origin. The result in (a) and (b) become : **(a)** $\overrightarrow{AF} = \frac{1}{3(1+r)}\overrightarrow{AB} + \frac{2}{3(1+r)}\overrightarrow{AC}$ **(b)** $\overrightarrow{AF} = \frac{1}{1+s}\overrightarrow{AB} + \frac{s}{3(1+s)}\overrightarrow{AC}$ where $\mathbf{a} = \mathbf{0}$

Now, we can compare these two results and get :

$\frac{1}{3(1+r)} = \frac{1}{1+s}$..........(1) $\quad \frac{2}{3(1+r)} = \frac{s}{3(1+s)}$......(2)

From which we can get : $r = \frac{4}{3}$, $s = \underline{\underline{6}}$